Distributed Graph Algorithms for Planar Networks

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CMU

joint work with Mohsen Ghaffari (MIT)

ADGA, Austin, October 12th 2014

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Distributed: CONGEST($\log n$) model

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Network Optimization Problems:

- shortest path (single/multiple source(s), (non-)negative weights, ...)
- network flows (min-cost, multi-commodity, ...)
- trees (MST, decomposition, steiner-tree, ...)
- TSP, min (st-)cut, facility location, ...

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- Many real-world networks/problems have planar like structure.
- Studying planar like networks led to:
  - a rich theory,
  - a powerful algorithmic toolbox, and
  - drastically improved algorithms.

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Goal:

Distributed toolbox/algorithms/theory for planar networks.

Algorithms in Planar vs. General Networks, e.g.:

Sequential Toolbox/Algorithms for Planar Networks

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- Much Faster Algorithms
  - SS-shortest-path: $O(n \log^2 n)$ vs. $O(mn)$ / $O(n)$ vs. $O(n \log n)$
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  - TSP: $3/2 - \epsilon$
  - multi-way cut: $O(\log k)$
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Toolbox:

- Planarity Testing / Embedding / Graph Drawing
- Decompositions (Separators, RS-decomp., tree width, ...)
- Bidimensionality, ...

Distributed Setting: Round Complexity in CONGEST

Trivial Bounds:

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General Graphs: Minimum Spanning Tree:

- $\tilde{O}(D + \sqrt{n})$ [KP'95]
- Strong $\tilde{\Omega}(\sqrt{n})$ Lower Bound [RP'99,E'04,DHK^{+}'11]
  - despite tiny diameter, e.g., $D = \log n$
  - even for any approximation
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$\tilde{O}(D)$ network optimization algorithms for planar networks.

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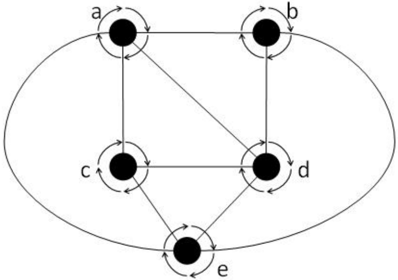
Goal:

Develop a new **distributed** algorithmic toolbox for planar networks.

New Tools I: A Distributed Planar Embedding Algorithm

An embedding is often necessary (not just knowing its existence).



Claim 1:

There is a $\tilde{O}(D)$ distributed planar embedding algorithm.

Efficient (Distributed) Planar Embedding Algorithm

General Idea [LEC'67,HT'08]

- Build embedding incrementally by adding vertices
- Track all possible partial embeddings

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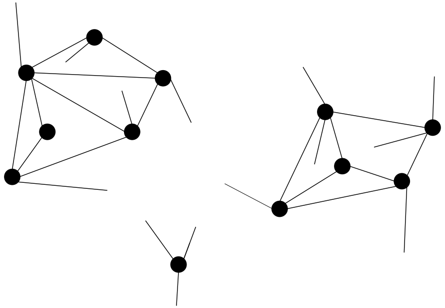
Partial Embedding:



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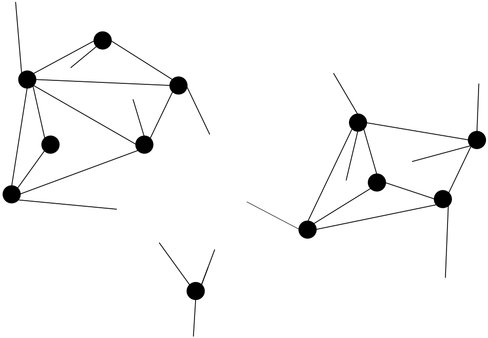

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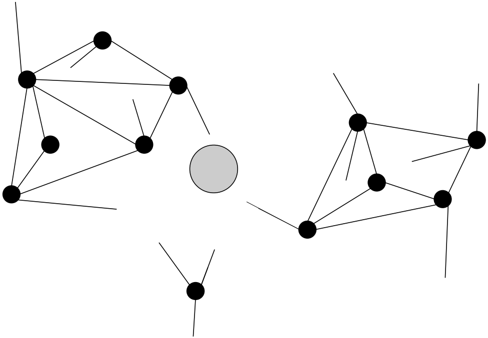

Choose the (parallel) embedding order such that the non-embedded vertices are connected.

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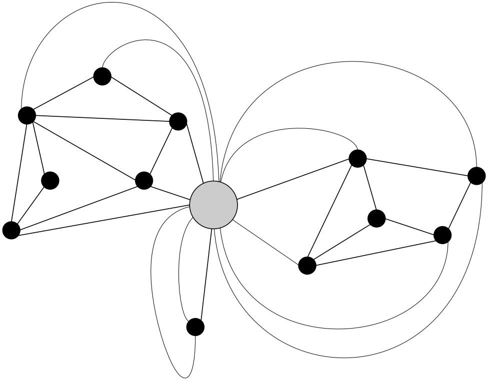

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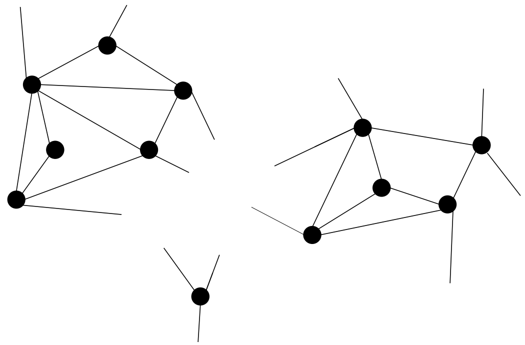

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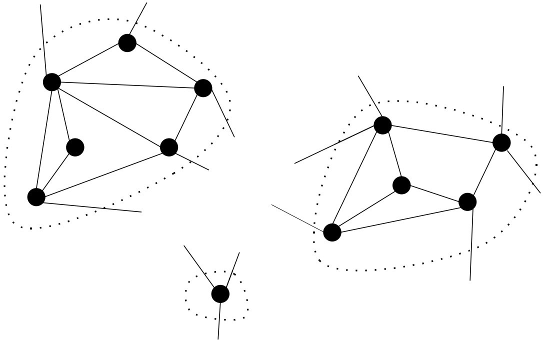

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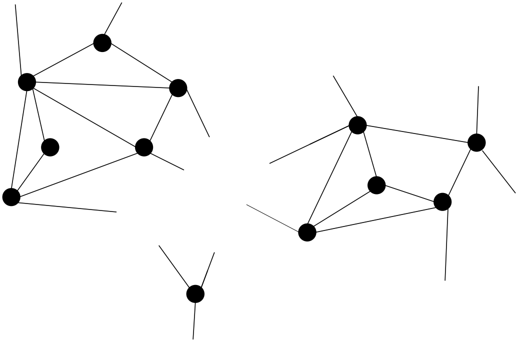
- all half-embedded edges lie in one face
- partial embedding: rotation of half-embedded edges around each connected component

Vertex Addition



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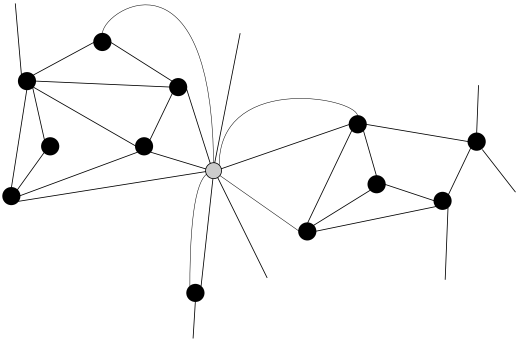

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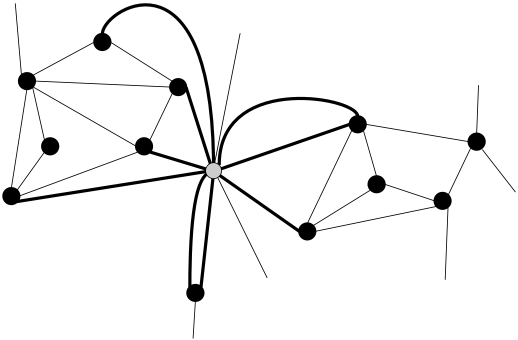

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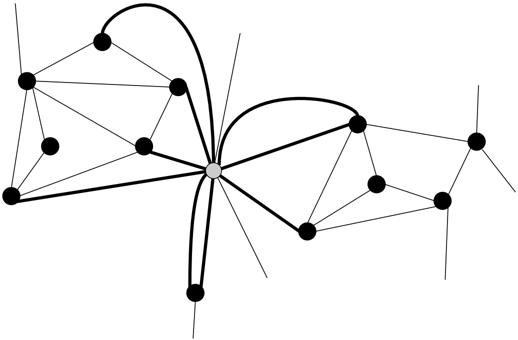

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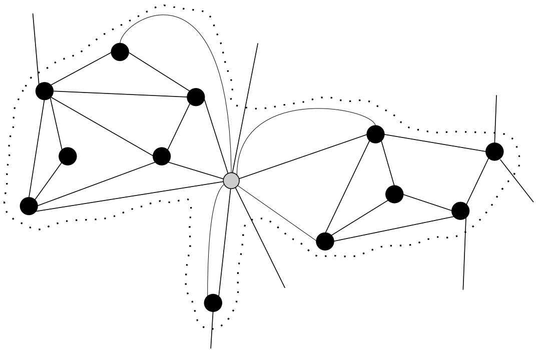

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  - flipping biconnected components
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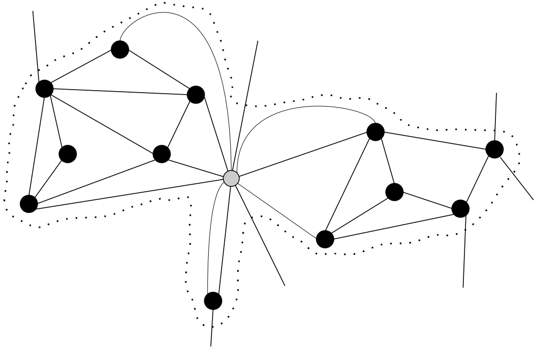

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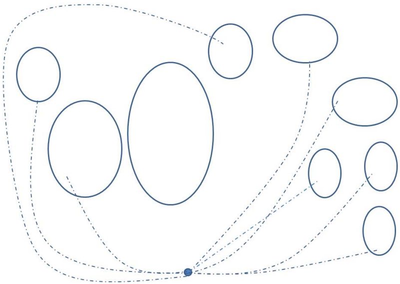
- recursively subdivide into balanced, low-diameter subproblems



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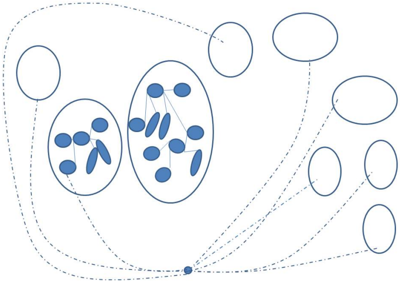

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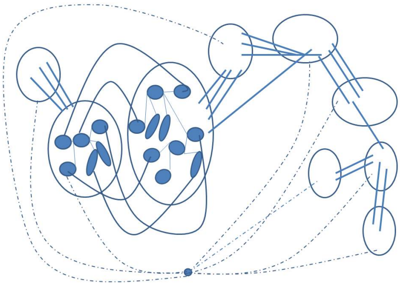

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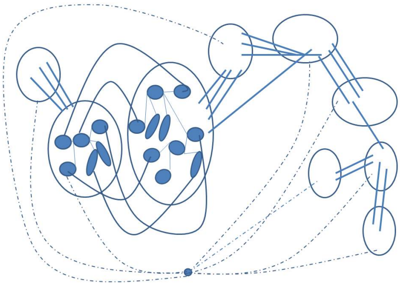

Claim 1:

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New Tools II: Low-Congestion Shortcuts

Uniquely Distributed Problem:

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Idea:

- Add extra edges to each subproblem to ensure low diameter.
- Avoid congestion, i.e., not use any edge too often

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Definition: c-congestion d-dilation shortcuts

Given planar $G = (V, E)$ and partition $S_1, \ldots, S_N \subset V$ with $G[S_i]$ connected. $H_1, \ldots, H_N \subset G$ are (c, d)-shortcuts iff:

(1) $\forall i$: diameter of $G[S_i] + H_i$ is at most d.

(2) Each edge $e \in E$ is in at most c of the subgraphs H_i.

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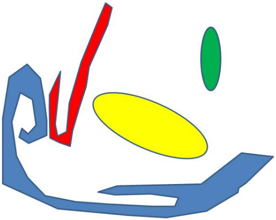
$(D \log D, D \log D)$-Shortcuts exist, can be computed distributedly in $\tilde{O}(D)$ rounds, and are essentially best possible.



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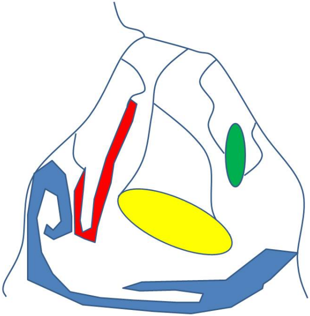

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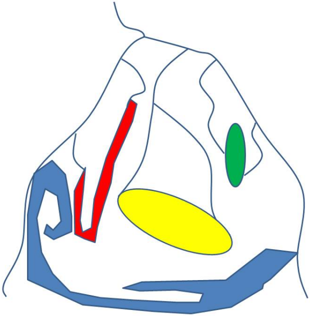

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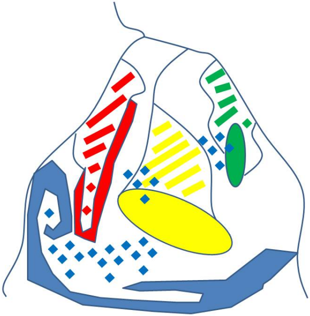

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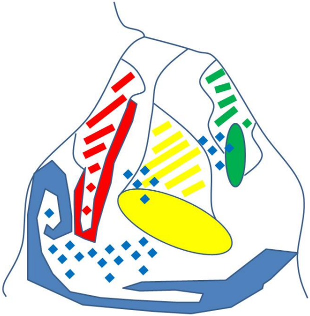

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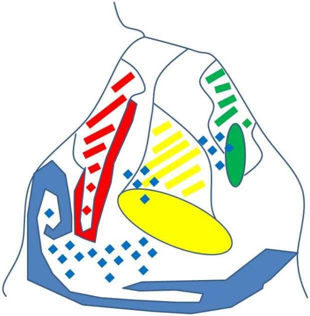

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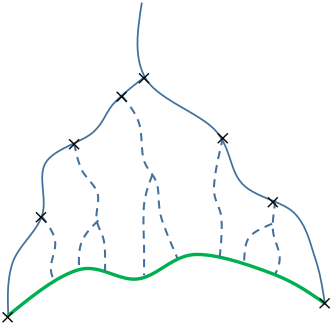

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- Dilation D^2:
  - Project any S_i-path P_i onto its T-path
  - Shortcut as far as possible within the enclosed subtree of T
  - At most D shortcuts each of lenght D needed

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Claim 3:

For planar networks there is a $\tilde{O}(D)$ distributed MST algorithm.

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    - compute short-cuts
    - compute $O(c + d) = O(D \log D)$ scheme
  - $\implies$ find new cheapest edge in $\tilde{O}(D)$ rounds

Summary and Open Questions

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Open Questions:

- Maximum Flow in $o(n)$
- Exact / Approximate Shortest-Paths in $\tilde{o}(\sqrt{n})$
- Depth-First-Search Trees in $o(n)$
- Separators (construction, useful definitions)
- Extensions, e.g., to bounded genus or excluded minor graph classes
- ...

Thank you!

Questions?